Lecture Notes on **Discrete Mathematics**.
Birzeit University, Palestine, 2020

First Order Logic

Mustafa Jarrar

3.1 Introduction to First Order Logic

3.2 Negation and Conditional Statements

3.3 Multiple Quantifiers in First Order Logic

1

1

Watch this lecture and download the slides

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Acknowledgement:
This lecture is based on (but not limited to) to chapter 3 in "Discrete Mathematics with Applications by Susanna S. Epp (3rd Edition)".

2

2

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First Order Logic

3.1 Introduction

In this lecture:

- ❑ Part 1: **What is a predicate, and Predicate Logic**
- ❑ Part 2: Universal and Existential Quantifiers: ∀, ∃
- ❑ Part 3: Formalize and Verablize Statements
- ❑ Part 4: Tarski's World (Simple Example)

3

3

First Order Logic

Is also called:

- The Logic of Quantified Statements
- Predicate Logic
- First-Order Predicate Calculus
- Lower Predicate Calculus
- Quantification theory

4

4

What is First Order Logic?

Propositional Logic

P, Q	*Propositions*
$\neg P$	Negation
$P \wedge Q$	Conjunction
$P \vee Q$	Disjunction
$P \rightarrow Q$	Implication
$P \leftrightarrow Q$	Equivalence

We regard the world as *Propositions*

First Order Logic

$P(x..y), Q(t,..s)$	*Predicates*
$\neg P$	Negation
$P \wedge Q$	Conjunction
$P \vee Q$	Disjunction
$P \rightarrow Q$	Implication
$P \leftrightarrow Q$	Equivalence
$\forall$	Universal quantification
$\exists$	*Existential* quantification

We regard the world as *Quantified Predicates*

5

5

What is a Predicate (محمول، صفة)

Definition

A **predicate** is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables. The **domain** of a predicate variable is the set of all values that may be substituted in place of the variable.

$$P(x_1,x_2 \ , \ldots , x_n)$$

Examples:

Person(Amjad),
University(BZU)
StudyAt(Amjad, BZU)

6

6

Arty of Predicates

$P(x_1, x_2, \ldots, x_n)$

Predicate Name — Arguments

Examples:

Unary Predicates:	Person(Amjad), University(BZU)
Binary Predicates:	StudyAt(Amjad, BZU)
Ternary Predicates	StudyAt(Amjad,BZU, CS)
Quaternary Predicate:	StudyAt(Amjad,BZU, CS, 2015)
n-ary Predicate:	StudyAt(Amjad,BZU, CS, 2015, BA, ….)

7

7

Truth of Predicates

What does predicate means exactly?

Definition

If $P(x)$ is a predicate and x has domain D, the **truth set** of $P(x)$ is the set of all elements of D that make $P(x)$ true when they are substituted for x. The truth set of $P(x)$ is denoted

$$\{x \in D \mid P(x)\}$$

Examples:

$\{x \in \mathit{Organization} \mid \mathit{University}(x)\}$

The set of all organizations that are universities.

$\{x \in \mathit{Person} \mid \mathit{student}(x)\}$

The set of all persons that are students.

8

8

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9

9

Universal Statements (∀)

القضايا الكلية

Definition

Let $Q(x)$ be a predicate and D the domain of x. A **universal statement** is a statement of the form "$\forall x \in D, Q(x)$." It is defined to be true if, and only if, $Q(x)$ is true for every x in D. It is defined to be false if, and only if, $Q(x)$ is false for at least one x in D. A value for x for which $Q(x)$ is false is called a **counterexample** to the universal statement.

Examples:

$\forall p \in \text{Palestinian} . \text{Likes}(p, \text{Zatar})$

$\forall x \in \mathbf{R} . x^2 \geq x$?

Let D= {1,2,3,4,5}
$\forall x \in D . x^2 \geq x$?

10

10

Existential Statements (∃)

القضايا الجزئية

Definition

Let $Q(x)$ be a predicate and D the domain of x. An **existential statement** is a statement of the form "$\exists x \in D$ such that $Q(x)$." It is defined to be true if, and only if, $Q(x)$ is true for at least one x in D. It is false if, and only if, $Q(x)$ is false for all x in D.

Examples:

$\exists p \in \text{Person} . \text{Likes}(p, \text{Jerusalem})$

$\exists m \in \mathbf{Z}^+ . m^2 = m$?

Let $E = \{5,6,7,8\}$?
$\exists m \in E . m^2 = m$

11

11

Universal vs. Existential Quantifiers

What ∀ and ∃ mean exactly?

$\forall x \in D . Q(x) \Leftrightarrow Q(x_1) \wedge Q(x_2) \wedge \ldots \wedge Q(x_n)$

$\exists x \in D . Q(x) \Leftrightarrow Q(x_1) \vee Q(x_2) \vee \ldots \vee Q(x_n)$

12

12

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13

13

Verbalizing Formal Statements

Write the following formal statements in an informal language:

$\forall x \in \mathbf{R} \cdot x^2 \geq 0$

The square of every real number equals or more than zero

$\forall x \in \mathbf{R} \cdot x^2 \neq -1$

The square of any real number does not equal -1

$\exists m \in \mathbf{Z}^+ \cdot m^2 = \mathrm{m}$

There is a positive integer equals its square

$\forall x \in \mathbf{R} \cdot x > 2 \rightarrow x^2 > 4$

If a real number more than 2 then is square is grater than 4

14

14

Formalize Statements

Write the following informal statements in a formal language:

All triangles have three sides
$\forall t \in Triangle \cdot \text{ThreeSided}(t)$

No dogs have wings
$\forall d \in Dog \cdot \neg \text{HasWings}(d)$

Some programs are structured
$\exists p \in \text{Program} \cdot \text{structured}(p)$

If a real number is an integer, then it is a rational number
$\forall n \in \text{RealNumber} \cdot \text{Integer}(n) \rightarrow \text{Rational}(n)$

All bytes have eight bits
$\forall b \in \text{Byte} \cdot \text{EightBits}(b)$

No fire trucks are green
$\forall t \in \text{FireTruck} \cdot \neg \text{Green}(t)$

15

15

Different Writings

$\forall x \in Square \ . \ \text{Rectangle}(x)$
$\forall x$. If x is a square then x is a rectangle
$\forall$ *Squares x . x is a* a rectangle

Although the book uses this notation but it's not recommended as predicates are not clear.

$\forall p \in \text{Palestinian} \ . \ \text{Likes}(p, \text{Jerusalem})$
$\forall p \ . \ \text{Palestinian}(p) \wedge \text{Likes}(p, \text{Jerusalem})$

$\exists p \in \text{Person} \ . \ \text{Likes}(p, \text{Jerusalem})$
$\exists p . \ \text{Person}(p) \wedge \text{Likes}(p, \text{Jerusalem})$

16

16

Quantifications might be Implicit

Formalize the following:

If a number is an integer, then it is a rational number
$\forall n \cdot \text{Integer}(n) \rightarrow \text{Rational}(n)$

If a person was born in Hebron then s/he is Khalili
$\forall x \in \text{Person} \cdot \text{BornInHebron}(x) \rightarrow \text{Khalili}(x)$
$\forall x \in \text{Person} \cdot \text{BornIn}(x, \textit{Hebron}) \rightarrow \text{Khalili}(x)$

People who like Homos are smart
$\forall x \in \text{Person} \cdot \text{Like}(x, \textit{Homos}) \rightarrow \text{Smart}(x)$
$\forall x \in \text{Person} \cdot \text{LikeHomos}(x) \rightarrow \text{Smart}(x)$

17

17

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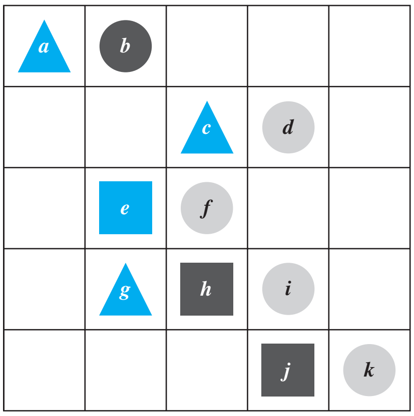
18

18

Tarski's World Example

The following statements are true or false?



$\forall t \,.\, \text{Triangle}(t) \rightarrow \text{Blue}(t)$ ✓

$\forall x \,.\, \text{Blue}(x) \rightarrow \text{Triangle}(x)$ ✗

$\exists y \,.\, \text{Square}(y) \wedge \text{RightOf}(d, y)$ ✓

$\exists z \,.\, \text{Square}(z) \wedge \text{Gray}(z)$ ✗